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Bayesian Analysis of Additive Chen-Weibull Distribution Using Square Error Loss Function

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Abstract*: The aim of this work was to study the Nipah virus (NiV) infection, a newly emerging zoonosis that causes severe disease in both animals and humans. Human population have been plagued by diseases of various types and origins. Zoonotic disease which have the capability of been transmitted from specie to another or to other animals. According to WHO, Nipah virus infection was first recognized in a large outbreak of 265 suspected cases in peninsular Malaysia during September 1998 to April 1999. These pathogens typically survive in a reservoir host. The lists of possible reservoir hosts capable of transmitting disease to humans are apes, insects, rodents, and bats. The diseases are then passed to humans who come in contact with an infected animal through bites or scratches, an infected animal's environment, or animal secretions such as saliva, faeces, or mucus. Nipah virus is an enveloped, negative-sense, single-stranded RNA virus in the family Paramyxoviridae, genus Henipavirus. The name of the virus and disease was from the village of "Sungai Nipah" in Malaysia where the first human cases lived. Nipah virus invade its host by is by inducing syncytial cell formation which spread rapidly through the vascular tissue of the infected host. Incubation time is usually short between 2 and 10 days. The Nipah virus primarily attacks the respiratory system, which is supported by the finding of high concentrations of viral antigens are found in the respiratory tract and lung epithelium.*

Keywords: *Nipah virus, Zoonotic disease, Animal, Host, Outbreak.*

INTRODUCTION

Many applications of survival and reliability engineering require lifetime models with bathtub-shaped failure rate functions. The Weibull distribution is one of the most commonly used lifetime models in reliability and survival analysis. However, the FR function of the Weibull distribution can only be increasing, constant, or decreasing. Therefore, the Weibull distribution is not capable of modeling lifetime data with a bathtub-shaped FR function. An additive Chen-Weibull distribution is a continuous lifetime model, proposed and studied by Tien and Radim (2020). The model was proposed by combining the Weibulland and Chen distributions together. This situation is of particular interest in applications where the failure time of a system with two or more failure modes must be modeled. One component, representing the first failure mode, follows the Chen distribution, and the other follows the Weibull distribution. Bayesian analysis is one of the most commonly used methods for estimating the unknown parameters of any probability distribution or model. Bayesian inference is a method of statistical inference in which the Bayes theorem is used to update the probability of a hypothesis as more evidence or information becomes available. Various researchers in the literature have used this method for estimating the unknown parameters of any proposed models. Example: Chris and Noor (2012) studied Bayesian analysis of the survival function and failure rate of the Weibull distribution with censored data. The Bayes estimator is obtained under three different loss functions. (Vikas et al., 2017) studied Bayesian and classical methods for estimating the parameter of the power Lindley (PL) distribution with application to waiting time data. Ahmad and Ahmad (2013) considered a Bayesian approach for the estimation of the scale parameter of a two-parameter Weibull distribution with a known shape. They obtained Baye's estimator of the Weibull distribution by using Jeffery's and an extension of Jeffery's prior under the linear exponential loss function and the symmetric loss function. Kamram et al. (2019) considered Bayesian analysis of a three-parameter Frechet distribution with medical applications. Al Omari (2016) studied Bayesian using the MCMC of the Gompertz distribution based on interval-censored data with three loss functions. Ahmad and Ahmad (2013) studied the Bayesian approach for two-parameter Weibull distributions using R software. Tien and Radim (2020) used Bayesian and classical estimation methods for estimating the four unknown parameters of the additive Chen Weibull distribution. In the Bayesian approach, a non-informative gamma prior under the square error loss function (SELF) was employed. The maximum likelihood (ML) methods of estimation were used for the classical approach. (Abbas et al., 2021) estimated the four unknown parameters of the additive Chen-Weibull (ACW) distribution using a Bayesian approach, considering the half-Cauchy prior distribution under the square error loss function (SELF) and the maximum product of spacing (MPS) method of estimation. The CDF of the additive Chen-Weibull (ACW) distribution with four parameters $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \lambda)'$ is defined by

$$F(x) = 1 - e^{\lambda(1-e^{x\gamma})-(\alpha x)^{\beta}}, \quad (1)$$

Where $x \geq 0, \alpha, \beta, \gamma > 0, \lambda \geq 0$

The probability density function (PDF) is defined by

$$f(x) = (\lambda\gamma x^{\gamma-1} e^{x\gamma} + \alpha\beta\left(\alpha x\right)^{\beta-1}\Big) e^{\lambda(1-e^{x\gamma})-(\alpha x)^{\beta}}, \quad (2)$$

Where $x \geq 0$

And the failure rate and reliability functions are respectively

$$h(x) = (\lambda\gamma x^{\gamma-1} e^{x\gamma} + \alpha\beta(\alpha x)^{\beta-1} \quad (3)$$

And

$$R(x) = e^{\lambda(1-e^{x\gamma})-(\alpha x)^{\beta}} \quad (4)$$

This article explores various Bayesian analyses for estimating four unknown parameters of the additive Chen-Weibull (ACW) distribution using exponential distribution prior under the square error loos function. The study presents the maximum product of the spacing method of estimation (MPSE) and Bayes estimators for estimating four unknown parameters of the ACW model, comparing their effectiveness using Kolmogorov-Smirnov test statistics.

MATERIAL AND METHODS

Estimation Using Maximum Product of Spacing Method

The discussion centered on the maximum product of the spacing method for estimating the parameters of any probability distribution. In this section, the method of maximum product of spacing (MPS) introduced by Chen and Amin (1979) will be used to estimate the unknown parameters of the additive Chen-Weibull (ACW) distribution. Let be i.i.d. random variables from the additive Chen-Weibull distribution, and let be the corresponding order statistics. The cumulative distribution function (CDF) of the additive Chen-Weibull (ACW) distribution with four parameters $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \lambda)^T$ is defined by

$$F(x) = 1 - e^{\lambda(1-e^{x\gamma})-(\alpha x)^\beta}, \quad x \geq 0, \alpha, \beta, \gamma > 0, \lambda \geq 0 \qquad (5)$$

Then, we define spacing as

$$D_i = F(X_i) = 1 - e^{\lambda(1-e^{x_i\gamma})-(\alpha x)^\beta} \qquad (6)$$

$$D_{n+i} = 1 - F(X_i) = 1 - 1 - e^{\lambda(1-e^{x_i\gamma})-(\alpha x)^\beta}$$

And the general term of spacing is given by

$$D_i = F(X_i) - F\left(X_{(i-1)}\right)$$

Such that $\sum_i^n D_i = 1$

In method of product spacing, we choose$\boldsymbol{\theta}$such that it maximizes the product of spacing or in other words it maximizes the geometric mean of spacing i.e.

$$M = \prod_{i=1}^{n+1} D_i^{\frac{1}{n+1}} \qquad (7)$$

We defined the term **S** which is obtained by taking log on both side of the equation (8) i.e Set $S = logM$ we get

$$S = \frac{1}{(n+1)} \sum_{1=1}^{n+1} logD_i$$

$$S = \frac{1}{(n+1)} \sum_{i=1}^{n+1} \log\left(1 - e^{\lambda(1-e^{x_i\gamma})-(\alpha x_i)^\beta} - 1 - e^{\lambda\left(1-e^{x_{(i-1)}\gamma}\right)-(\alpha x_{(i-1)})^\beta}\right)$$

$$S = \frac{1}{(n+1)} \sum_{i+1}^{n+1} \lambda(1 - e^{x_i\gamma}) - (\alpha x_i)^\beta - \lambda(1 - e^{(x_{i-1})^\gamma})\alpha(x_{i-1})^\beta \qquad (8)$$

To obtain the normal equations for the unknown parameters, we differentiate partially equation (8) with respect to the four (4) parameters (α, β, γ and λ) and equate them to zero. The estimators for α, β, γ and λ can be obtained by

$$\frac{ds}{d\alpha} = -\frac{1}{(n+1)} \sum_{i=1}^{n+1} (x_i)^\beta + \lambda - \lambda e^{(x_{i-1})^\gamma} (x_{i-1})^\beta = 0$$

$$\frac{ds}{d\beta} = -\frac{1}{(n+1)} \sum_{i=1}^{n+1} (\alpha x_i)^\beta - \lambda - \lambda e^{(x_{i-1})^\gamma} \alpha (x_{i-1})^{\beta-1} = 0$$

$$\frac{ds}{d\gamma} = -\frac{1}{(n+1)} \sum_{i=1}^{n+1} x_i \lambda e^{x_i\gamma} + \alpha(x_{i-1})^\beta \alpha(x_{i-1}) e^{(x_{i-1})\gamma} = 0$$

$$\frac{ds}{d\lambda} = -\frac{1}{(n+1)} \sum_{i=1}^{n+1} 1 - e^{x_i\gamma} + \alpha(x_{i-1})^\beta (1 - e^{(x_{i-1})})^\gamma = 0$$

The above expressions cannot be solve analytically, therefore, the iterative procedure techniques (conjugate-gradient algorithm solution) will be used in order to obtain the estimate of the parameters of ACW distributions.

Bayesian Estimation Under Exponential Prior

Suppose $X: x_1, x_2, \dots x_n$ be a random sample from the additive Chen-Weibull (ACW) distribution with probability density function (PDF)

$$f(x) = (\lambda\gamma x^{\gamma-1} e^{x\gamma} + \alpha\beta(\alpha x)^{\beta-1}) e^{\lambda(1-e^{x\gamma})-(\alpha x)^{\beta}}, \qquad x \geq 0.$$

The corresponding likelihood function can be defined as

$$L(X|\alpha,\beta,\gamma,\lambda) =$$

$$[\prod_{i=1}^{n} \left(\lambda\gamma x_i^{\gamma-1} e^{x_i^{\gamma}} + \alpha\beta(\alpha x_i)^{\beta-1}\right)] \exp\left[-\sum_{i=1}^{n} (\lambda(1 - e_i^{x^{\gamma}}) + (\alpha x_i)^{\beta-1})\right] \qquad (9)$$

And the log-likelihood is defined as

$$\log L(Y|\alpha,\beta,\gamma,\lambda) = \gamma - 1 \sum_{i=1}^{n} \log(\lambda\gamma x_i) + x_i^{\gamma} + \beta - 1 \log(\alpha\beta) + \log(\alpha x_i) - \sum_{i=1}^{n} (\lambda(1 - e^{x_i^{\gamma}}) + \beta \log(\alpha x_i) \qquad (10)$$

In this section we consider the Bayesian estimation of the unknown parameters of additive Chen-Weibull (ACW) distribution parameters (α, β, γ and λ). The Bayes estimate is considered under the assumption that a random variables (α, β, γ and λ) have an independent exponential prior distribution. Assumed that $\alpha \sim exponential\ (\delta_{1)}$, $\beta \sim exponential\ (\delta_{2)}\gamma \sim exponential\ (\delta_3)$ and $\lambda \sim exponential\ (\delta_4)$.

Where $\delta_{1,}\delta_2, \delta_3$ and δ_4 are the hypeparameters.

Then the prior density of each four (4) unknown parameters can be given as

$$P(\alpha) = \delta_1, e^{\delta_1 \alpha} \delta_1 > 0 \; P(\beta) = \delta_2 e^{\delta_2 \beta}, \delta_2 > 0 P(\gamma) = \delta_3 e^{\delta_3 \gamma,} \delta_3 > 0 P(\lambda) = \delta_4 e^{\delta_4 \lambda} \delta_4 > 0$$

Then joint prior density function of the four parameters can be written as follows:

$$P(\alpha,\beta,\gamma,\lambda) \propto e^{\delta_1\alpha + \delta_2\beta + \delta_3\gamma + \delta_4\lambda}$$

Based on the likelihood function above, and by Bayes' rules, the joint posterior distribution of α, β, γ and λ given the observed data X is defined as

$$P(\alpha,\beta,\gamma,\lambda|X) \propto L(X|\alpha,\beta,\gamma,\lambda) P(\alpha,\beta,\gamma,\lambda) \qquad (11)$$

Taking the log of the prior densities, the logarithm of the unnormalized joint posterior density is calculated according to the Bayes' rule as:

$$\log p\, P(\alpha,\beta,\gamma,\lambda|X) \propto \log L\,(X|\alpha,\beta,\gamma,\lambda) + \log p\left(e^{\delta_1\alpha}\right) + \log p(e^{\delta_2\beta}) + \log p(e^{\delta_3\gamma}) + \log p(e^{\delta_4\lambda}) \qquad (12)$$

To get the correct posterior inference for the positive parameters in the situation that involves optimization of the log-posterior, itself a difficult numerical problem. The package LaplacesDemonfavour unconstrained parameterization by making the log-transformation of the positive parameter. In fact, working on log scale make computation numerically more stable (Shehla, 2016).

Estimation under square error loss function

The Bayesian estimates of the four parameters of additive Chen-Weibull (ACW) distribution assuming independent exponential prior under square error loss function (SELF) is given by

$$\hat{\alpha} = \int \alpha P(\gamma, \alpha, \lambda, \beta | X) d\theta \, \hat{\beta} = \int \beta p((\gamma, \alpha, \lambda, \beta | X) \, d\theta \qquad (13)$$
$$\hat{\gamma} = \int \gamma p((\gamma, \alpha, \lambda, \beta | X) \, d\theta \hat{\lambda} = \int \lambda p((\gamma, \alpha, \lambda, \beta | X) \, d\theta \qquad (14)$$

As we can see from the above expressions that, the Bayes estimator cannot be analytically computed through the posterior means. Therefore, Laplace approximation and Monte Carlo Markov Chain (MCMC) will be used to approximate the posterior densities of the four unknown parameters of additive Chen-Weibull distribution.

The influence of prior distribution on posterior inference decreases as sample size n increase. These ideas are sometimes referred to as asymptotic theory. The large sample results are not actually necessary for performing Bayesian data analysis but are often useful for quick references and as starting points for iterative simulation algorithms (Gelman *et al.,* 2004). A remarkable method of asymptotic approximation is the Laplace approximation (Tierney, 1986 &1989) which currently approximates the unimodal posterior moments and marginal posterior densities in many cases. A brief and informal description of Laplace approximation method is as follows:

Suppose $-h(\theta)$ is a smooth, bounded and unimodal function with a maximum at $\hat{\theta}$ where θ is a scalar and we wish to evaluate the integral

$$1 = \int q(\theta) \exp\big(-nh(\theta)\big) d\theta, \theta \in \Theta \qquad (15)$$

As presented by [22], Laplace's method involves the Taylor's series expansion of q and h about$\hat{\theta}$. As $h'\big(\hat{\theta}\big) = 0$, it follows that

$$h(\theta) = h\big(\hat{\theta}\big) + \big(\theta - \hat{\theta}\big)' h'\big(\hat{\theta}\big) + \frac{1}{2}(\theta - \hat{\theta})^2 h''\big(\hat{\theta}\big) + \cdots$$
$$= h\big(\hat{\theta}\big) + \tfrac{1}{2} \, (\theta - \hat{\theta})^2 h''\big(\hat{\theta}\big) + \cdots \qquad (16)$$
$$q(\theta) = q\big(\hat{\theta}\big) + \frac{1}{2}(\theta - \hat{\theta})^2 q''\big(\hat{\theta}\big) + \cdots$$
$$I \approx 2(\pi)^{1/2} n^{-1/2} \sigma q(\hat{\theta}) \exp[-nh\big(\hat{\theta}\big)] \qquad (17)$$

Where $\sigma = [\frac{\partial^2 h}{\partial \theta^2} |_{\hat{\theta}}]^{-1/2}$

To calculate moments of the posterior distributions, we need to evaluate expression such as:

$$E\{g(\theta) = \frac{\int g(\theta) \exp\{-nh(\theta)\} d\theta}{\int \exp\{-nh(\theta)\} d\theta} \qquad (18)$$

Where $\exp\{-nh(\theta)\} = L(\theta | y) p(\theta)$by [22]

Upon applying (17) to both the numerator and denominator of (18) separately (with $q =$ to g and $q = 1$), a first order approximation

$$E\{g(\theta)\} = g(\hat{\theta})\{1 + 0(n^{-1})\}$$

Easily emerges. Thus, the Laplace approximation is of order $0(n^{-1})$ uniformly on any neighborhood of the mode. This means that it should provide a good approximation in the tails of the distribution also (Tierney 1989).

The independent Metropolis-Hasting algorithm is a general MCMC algorithm introduced by Hasting (1970) will be used to simulate a random sample from the posterior distribution. The implementation of independent Metropolis-Hasting algorithm and Laplace approximation are given below. Let as assume a target distribution $\boldsymbol{p}(\boldsymbol{\theta}|\boldsymbol{y})$from which we wish to generate a sample of size T. the metropolis-Hastings algorithm can be described by the following iterative steps; where $\boldsymbol{\theta}^{(t)}$ is the vector of generated values in t iteration of the algorithm.

Algorithm 1

Given the marginal posterior distribution $P((\gamma|\alpha,\lambda,\beta,Y)$,
$P((\alpha|\gamma,\lambda,\beta,Y)$,$P((\beta|\gamma,\alpha,\lambda,Y)$, $P((\lambda|\gamma,\alpha,\beta,Y)$ and sample size N:
Step 1: select a starting value of the chain$\gamma^{(0)},\alpha^{(0)},\beta^{(0)},\lambda^{(0)}$.
Step 2: set $m=1$.
Step 3: Using the M-H, generate $\gamma^{(m)}$ from $P((\gamma\big|\alpha^{(m-1)},\lambda^{(m-1)},\beta^{(m-1)}Y\big)$.
Step 4: Using the M-H, generate $\alpha^{(m)}$ from $P\big(\alpha\big|\gamma^{(m)},\lambda^{(m-1)},\beta^{(m-1)},Y\big)$.
Step 5: Using the M-H generate $\alpha^{(m)}$ from$P\big(\beta\big|\gamma^{(m)},\alpha^{(m)},\lambda^{(m-1)},Y\big)$.
Step 6: Using the M-H generate $\lambda^{(m)}$ from$P((\lambda\big|\gamma^{(m)},\alpha^{(m)},\beta^{(m)},Y\big)$.
Step 7: set $m=m+1$.
Step 8: Repeat step 2 to 7until $m=N$ to obtain the samples of γ,α,λ and β
With size$N$, respectively.

DATA

A two real data set will be used for illustration purpose that is; Aarset and Meeker-Escobar data with a random sample of 50 lifetime's devices and the failure and running times of 30 devices respectively. Two failure modes were observed for these data.

RESULT AND DISCUSSION

This section compares Bayesian and MPSE estimation methods of the additive Chen-Weibull (ACW) distribution on two real reliability data sets, using the Kolmogorov-Smirnov test for comparison.

Meeker-Escobar data

Table 1. Meeker-Escobar data

2	10	13	23	23	28	30	65	80	88
106	143	147	173	181	212	245	247	261	266
275	293	300	300	300	300	300	300	300	300

Table 1. Represent the Meeker-Escobar data with a failure and running times of 30 devices. Two failure modes were observed for this dataset. Many authors in the literature used this data for illustration purpose and the most recent studies are given by (Tien and Radim, 2020), Almaliki et al., (2013), Bo He et al., (2016) and Mohammed et al., (2019).

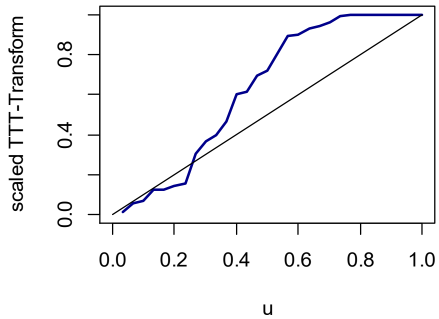


Figure 1. The empirical scaled TTT-transform plot for Meeker-Escobar data
Source: Author's computation aided by R package V 3.6.3

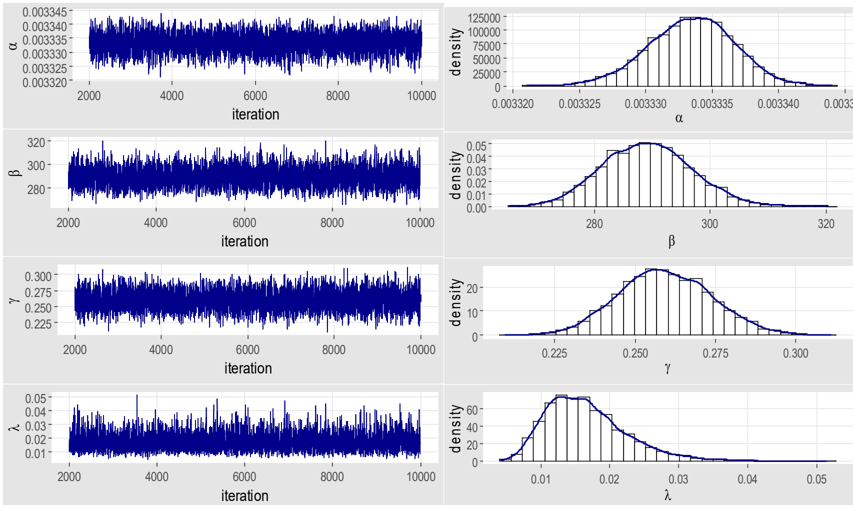
Fig. 1. Provides the empirical scaled TTT-transform plot for Meeker-Escobar data sets. From this plot we can observed that the Meeker-Escobar data have a bathtub-shaped failure rate function.The independent Metropolis-Hasting algorithm (algorithm 1) in MCMC is used to simulate a random sample from each of the marginal posterior density of four unknown parameters to approximate the posterior distribution of ACW model.

Table 2. Bayes estimates (assuming exponential prior) for the parameters When fitting ACW model to Meeker-Escobar dataset

Parameters	Bayes	SD	Median	Bayes 95%C.I
γ	0.25985	0.01452	0.25957	[0.23221, 0.28838]
α	0.00333	0.00000	0.00333	[0.00332, 0.00334]
λ	0.01626	0.00590	0.01542	[0.00758, 0.03018]
β	290.511	7.84002	290.318	[276.144, 306.402]

Source: Author's computation aided by R package V 3.6.3

Table 2 Shows Bayes estimates, Bayes 95% CI and standard deviation for ($\alpha, \beta, \gamma,$ andλ). Additionally, the asymptotic approximation method (Laplace approximation) is also used to simulate a random sample from the each of the marginal posterior density using sampling important resampling and approximate the posterior densities of the four parameters of the ACW model. The estimates of the four parameters ($\gamma, \alpha, \beta,$ and λ) by asymptotic approximation method are respectively computed as, the Bayes estimates are0.2572, 0.00333, 0.01626, and 290.511. The standard deviation of the four parameters are given as0.01452, 0.00000, 0.00590, and 7.84002. The Bayes 95% C.Is are also computed as[0.23221, 0.28838], [0.00332, 0.00334], [0.00758, 0.03018] and [276.144, 306.402].Therefore, from the result, it is cleared that the metropolis Hasting algorism in Monte Carlo Markov Chain (MCMC) technic summarizes the posterior more precisely in terms of the lower standards deviations of the parameters as compared to that of asymptotic approximation.



(A) **(B)**

Figure 2. (A) Trace plots and (B) plots of the marginal posterior densities of the parameters for the posterior distribution of additive Chen-Weibull model using the IM

Source: Author's computation aided by R package V 3.6.3

Fig. 2 (A) trace plot and (B) the marginal posterior density estimates of the parameters obtained by MCMC algorithm (algorithm 1). The trace plots of each parameter showed that IM algorithm converges quickly to the same target distribution. The marginal posterior densities of the four parameters are distributed approximately symmetrically around the central values which means that they provide good Bayesian estimates under square error loss function.

Table 3: MPSE of ACW model Using Meeker-Escobar data

Parameter	MPSEs
γ	0.15168
α	0.00333
λ	0.01518
β	13.4583

Source: Author's computation aided by R package V 3.6.3

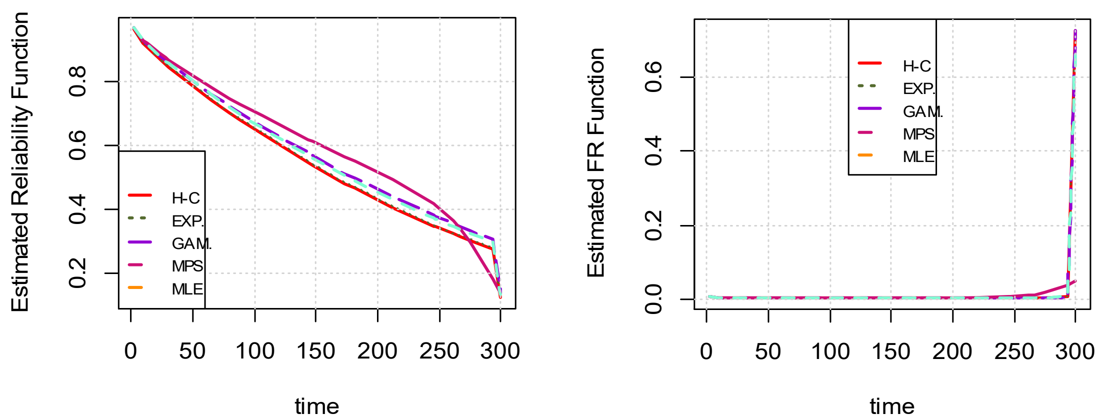
Table 3 gives the maximum product of spacing (MPS) point estimate of the four unknown parameters of additive Chen-Weibull (ACW) model of meeker-Escobar data using mpedist function in BMT package in R with good set of initial values of the parameters.

Table 4. K-S and its p-value when fitting to ACWTo Meeker-Escobar data

Method of Estimation	K-S(p-value)
Bayesian (Assuming exponential prior)	0.13908(0.6074)
Bayesian (Assuming Half-Cauchy prior)	0.14084(0.5912)
Bayesian (Assuming Gamma prior)	0.14054(0.594)
MPSEs	0.13729(0.6238)
MLEs	0.13423(0.6521)

Source: Author's computation aided by R package V 3.6.3

Table 4 represents the K-S statistic and its p-value for the comparison of the two different prior distribution (exponential and Half-Cauchy) for the estimation of four unknown parameters of ACW model to Meeker-Escobar dataset.The methods that produced highest p-value will be considered as the best methods of estimating the ACW model parameters. The result from K-S statistic showed that our prior distribution (exponential) with KS = 0.13908 with its corresponding p-value = 0.6074 perform better than the Bayesian approach assuming Half-Cauchy and gamma prior distributions.However, theMPSE and MLE with p-values 0.6238and 0.6521 respectively perform relatively equal and better than the Bayesian estimates assuming both exponential, Half-Cauchy and gamma prior distribution.

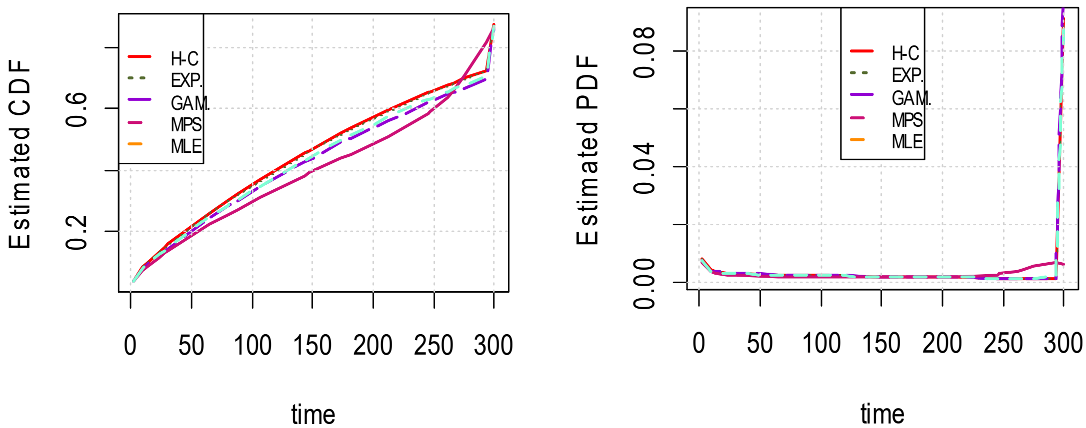


(A) **(B)**

Figure 3. The estimated (A) reliability function and (B) failure rate function obtained by fitting ACW distribution using Bayesian and MPS method of estimation to meeker-Escobar data

Source: Author's computation aided by R package V 3.6.3

Fig 3 showed a visual comparison of reliability (R) function and FR functions plots of the Meeker-Escobar data with fitted parameter values of Bayesian estimate (assuming three different prior) and two different classical methods of estimation.we can see from (B) that, the FR of Bayesianestimate assuming exponential prior started to increase at around $x = 300$, which shows a low and long constant FR at mid time region.



(A) (B)

Fig. 4 showed (A) the cumulative distribution function (CDF) and (B) the probability distribution function (PDF) of the Bayesian estimation method assuming three different prior and two classical methods. From the plots we can see that all the methods have different shapes. From the results obtained we can conclude that, in Bayesian estimate exponential prior provides a suitable fit to the Meeker-Escobar dataset. However, in classical methods the MPSEs provides a better fit of the four unknown parameters of ACW model.

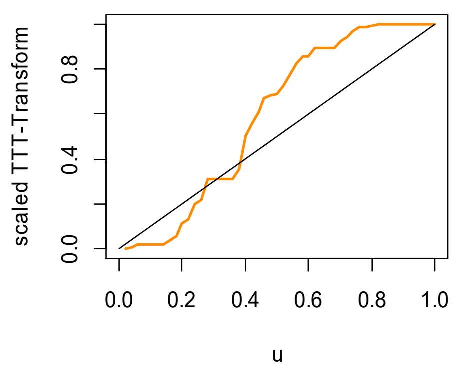


Figure 4. The empirical scaled TTT-transform plot for Aarset data
Source: Author's computation aided by R package V 3.6.3

Table 4. Bayes estimates (assuming exponential prior) for the parameters when Fitting ACW model to Aarset data.

Parameters	Bayes	SD	Median	Bayes 95%C.I
γ	0.2841	0.0147	0.2839	[0.2559, 0.3131]
α	0.0118	0.0000	0.0118	[0.0117, 0.0118]
λ	0.0418	0.0094	0.0409	[0.0262, 0.0626]
β	86.699	14.187	85.562	[62.009, 117.83]

Source: Author's computation aided by R package V 3.6.3

Table 4 Shows Bayes estimates, Bayes 95% CI and standard deviation for (α, β, γ, andλ). Additionally, the Laplace approximation is also considered to simulate a random sample from the each of the marginal posterior density using sampling important resampling and approximate the posterior densities of the ACW model's parameter. The estimates of the four parameters (γ, α, β, and λ) by asymptotic approximation method are respectively computed as, the Bayes estimates are0.2841, 0.0118, 0.0418, and 86.699. The standard deviation of the four parameters are given as0.0147, 0.00000, 0.0094, and 14.187. The Bayes 95% C.Is are also computed as[0.2559, 0.3131], [0.0117, 0.0118], [0.0262, 0.0626] and [62.009, 117.83]. Therefore, from the result, it is cleared that the metropolis Hasting algorism in Monte Carlo Markov Chain (MCMC) technic summarizes the posterior more precisely in terms of the lower standards deviations of the parameters as compared to that of asymptotic approximation.

Table 6: Parameter estimate of ACW model of MPSE method using Aarset dataset

Parameter	MPSEs
γ	0.2759
α	0.0118
λ	0.0423
β	38.231

Source: Author's computation aided by R package V 3.6.3

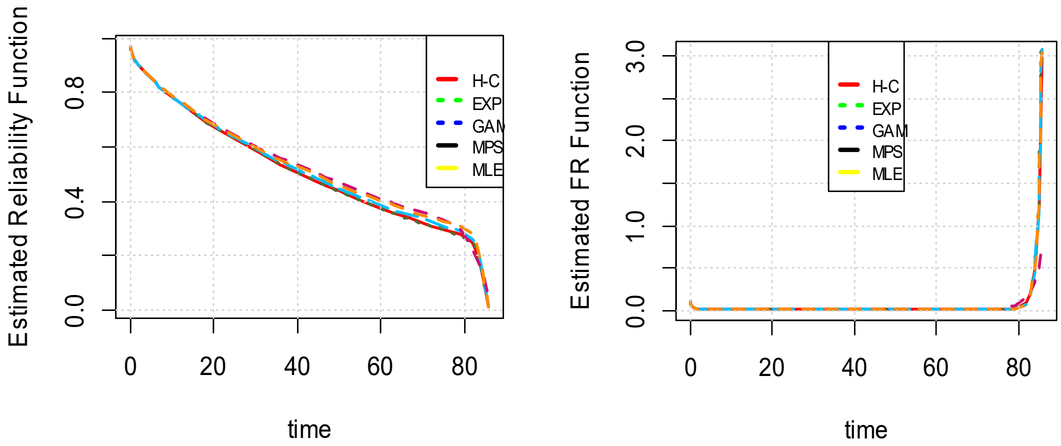
Table 6 gives the maximum product of spacing (MPS) point estimate of the four unknown parameters of additive Chen-Weibull (ACW) model ofAarset data using mpdist function in BMT package in R with good set of initial values of the parameters.

Table 8. K-S and its p-value for comparison of Bayesian and non-Bayesian Approach when fitting to Aarset data

Method of Estimation	K-S statistics	p-value
Bayesian (Assuming Exponential prior)	0.089631	0.8166
Bayesian (Assuming Half-Cauchy prior)	0.089235	0.8208
Bayesian (Assuming Gamma prior)	0.079682	0.9087
MPSEs	0.069895	0.9675
MLEs	0.070375	0.9654

Source: Author's computation aided by R package V 3.6.3

Table 8 represents the K-S statistic and its p-value for the comparison of the Bayesian approach assuming two different prior's distribution when fitting ACW model to Aarset data. The result from K-S statistic showed that, the Bayesian assuming exponential and Half-Cauchy performed relatively equal as there is slight difference in their K-S statistic and their corresponding p-values.Therefore, the MPSE with K-S = 0.069895 with its corresponding p-value=0.9675 and MLE with K-S = 0.070375 with its corresponding p-value = 0.9654Performed relatively equal and better than the Bayesian estimate assuming three differentpriordistribution.

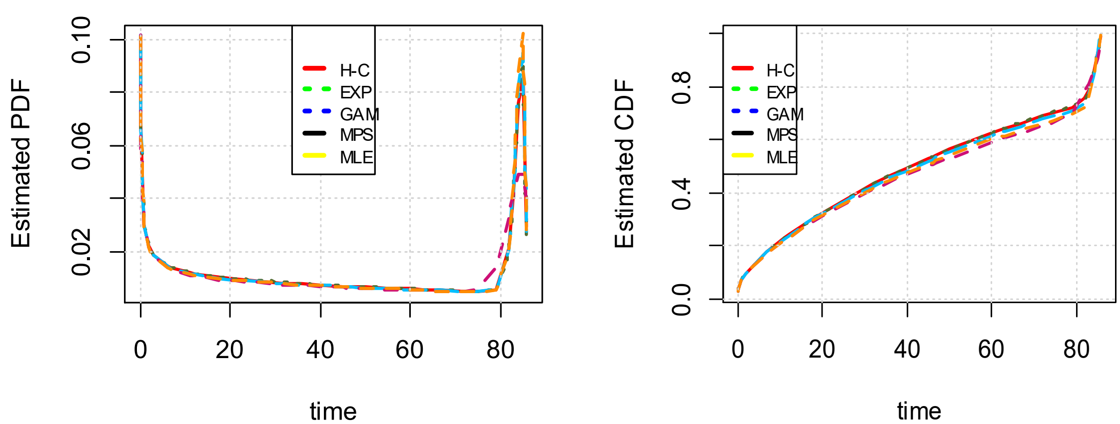


(A) **(B)**

Figure. 8. The estimated (A) reliability function and (B) failure rate function.

Source: Author's computation aided by R package V 3.6.3

Fig 8. Showed a visual comparison of survival function and failure rate functions obtained by fitting ACW distribution using Bayesian assuming three different prior distribution and two classical methods of estimation to Aarset data. it has been cleared from these plots that, the FR of both Bayesian and classical estimate started to increases at $x = 80$which show a relatively low and long constant FR at mid time region.



(A) **(B)**

Figure. 9. The estimated (A) PDF and (B) CDF of the ACW model

Source: Author's computation aided by R package V 3.6.3

Fig. 9 showed the plots of (A) cumulative distribution function (CDF) with corresponding (B) probability density function (CDF) plotsobtained by fitting ACW distribution using Bayesian assuming three different prior and classical method of estimation usingAarset data. We can observed from these plots that, the three different priorand two classical estimation methods have almost similar shape as there is only slight difference.

CONCLUSION

In this study, Bayesian inference assuming exponential prior distribution under square error loss function (SELF) is considered in estimating the four unknown parameters of the additive Chen-Weibull (ACW) model. Two real data sets were used for illustration purposes. In the Bayesian paradigm, the analytic approximation and MCMC techniques were implemented using the functions LaplaceApproximation and LaplacesDemon, respectively. Therefore, it has been observed throughout that the simulation technique, particularly the independent metropolis algorithm, summarizes the posterior more precisely in terms of the lower standard deviations of the parameters as compared to the Laplace approximation. From the results obtained, we can conclude that the Bayesian estimate assuming an exponential prior distribution performed better than the Bayesian estimate assuming a half-Cauchy prior distribution studied by Abbas et al. (2021).

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