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# Analytic Solution of Two – Dimensional Rectangular Heat Equation Using Neumann Boundary Value Conditions

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**Abstract: Abstract:** *In this paper, two-dimensional heat equation with Neumann boundary condition is presented and separation of variable of partial differential equations, double Fourier series and* residue theory *was employed for solving the problems. The obtained result provided exact analytical solution of the cosine function. However, the method is easier to implement. In case one (1) , two (2) and three (3) the results obtained using residue theory is the same as the other methods of solving ordinary differential equation, in case one (1) and two (2) the analytic solutions are trivial, while case three (3) the solutions is not. It is found that the technique is a powerful mathematical tool which can be extended to sciences, engineering, and Technology problems.*

**Keywords:** *Analytic solution, two-dimensional, Heat equation, Neumann, Fourier, boundary value*

## 1. Introduction

Joseph Fourier a French mathematician and physicist best known for initiating the study of Fourier series with applications to the theory of oscillating system and heat transfer (Tracy, 2017).

The study of heat equations refers to the transport of energy in a medium due to the temperature gradient (de Assis, 2017).

The theory of heat equations was first developed by Joseph Fourier in 1822; Heat is the dynamic energy of particles that are being exchanged and is connected with the study of Brownian motion. The one, two and three-dimensional wave equation was discovered by Alembert and Euler. The solutions of heat and wave equations have attracted the attention of various authors in mathematics, such as the optimal Homotopy asymptotic method (OHAM), the modified Adomian decomposition method (MADM), the variational iteration method, the differential transform method (DTM), the Homotopy perturbation method (HPM) (Hassan *et al.,* 2019)

Homotopy Perturbation Method (HPM) for solving one-dimensional heat conduction problem with Dirichlet and Neumann boundary conditions was proposed, the study reveals that more accurate result was obtained in the numerical techniques. The work is been extended in finding analytical Techniques two-dimensional heat equation with only Neumann boundary condition (Cheniguel, 2014).

In the field of engineering, analytical techniques can be used to classic heat transfer problems with regular geometric boundaries, analytical method it's difficult to apply in solving the mathematical and physical equations in the non-orthogonal boundary of irregular domains. Analytical method can be applied in the separation of variable, the integral transformation method, Green's functions method and the conformal mapping method, and the analytical method it's hard to solve in the irregular domains and it is easier to analyze a different physical phenomenon and it`s less time consuming compared with the numerical method (Fan. *et al.*, 2013).

Cortazar *et al.* (2006), Propose that the solution of the common Neumann boundary condition for the heat equation can be approximated by a solution of a sequence of nonlocal Neumann boundary value problems, the studied also reveals that the solution of the family of problems converge to a solution of the heat equation with Neumann boundary conditions.

Studied of Finite volume method was used to solve the transient partial differential equation for Heat transfer in two dimensions with initial and boundary conditions of mixed Dirichlet in a rectangular field and the numerical result reveals that the solution is exact (Hasnat *et al,* 2015).

Saeed 2015 investigate the analytical and numerical solution of one-Dimensional a rectangular Fin with an Additional Heat source. The study also investigated the influence of the heat source on the temperature profiles and the fin efficiency was discussed in the case of the heat source $\psi = 0$ for the wide range of parameters (M).

Heat conduction phenomena with appropriate boundary conditions occur often in several areas of science and engineering (Kainat. *et al.*, 2009)

Ivanchenko presents an analytical solution for the convection-diffusion problem in a cylindrical domain, with the use of separation of variable method in the polar coordinate system.

Hoshan (2014) Present Greens and Dual integral equations for solving inhomogeneous two-dimensional heat equations in cylindrical, spherical and other coordinate systems and the result is obtained with the method of Greens and Dual integral equations.

Power-series-expansion technique to solve approximately the two-dimensional wave equation, the technique is useful for a finite body of certain shape geometry. Recurrent formulas for the wave polynomials and their derivatives are obtained in the Cartesian and polar coordinate systems. The result is the derivation of the formulas for the wave polynomials that satisfying a heat equation and their derivatives. The Method is a straightforward for solving heat equations in a finite bodies, and it is also useful when the shape of the body is complicated, both in Cartesian and polar coordinates the result obtained show that the obtained approximate exact solutions are very good (ARTUR 2005).

Method of superposition and separation variables is applied to improve analytical solutions to the transient heat conduction for a two-dimensional cylindrical fin. The temperature distributions are generalized for a linear combination of the product of Bessel function, Fourier series and exponential type for nine different cases. The solutions can be used to prove the two- or three-dimensional. (Ko-Ta , *et al*, 2009).

Studied of two-dimensional heat transfer problems in cylindrical coordinates using finite difference method, the study also reveals that the finite difference method is effective in solving the problem of heat transfer in cylindrical coordinate (Mori, 2015).

Analytical and numerical methods were used in the correlation of the solution of two-dimensional steady-state heat conduction, and the result obtained shows that the finite element are in good agreement with the analytical values and solution reveals that it can be effectively used to more complex thermal problems (Suresh, 2018).

Thus the main goal of this paper is to apply the method of separation of variable, double Fourier series and Residue theory for two-dimensional heat equation problem with Neumann boundary condition.

## 2. Methodology

**2-Dimensional heat equation**

**Cartesian coordinate**

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial y^2}\right)$$ We put $T(x, y, t) = X(x)Y(y)\tau(t)$

$$\frac{1}{X}\frac{d^2X}{dx} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{\kappa\tau}\frac{d\tau}{dt} = -\mu^2 (say)$$ Where $\mu^2$ is separation constant, $\mu$ be real

Then $\tau = A_1 e^{-\kappa\mu^2 t}$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\left(\frac{1}{Y}\frac{d^2Y}{dy^2} + \mu^2\right) = -p^2 (say)$$

$$\frac{d^2X}{dx^2} + p^2X = 0 \ \ and \ \ \frac{d^2Y}{dy^2} + q^2Y = 0$$

$X = A\cos px + B\sin px$ *and* $Y = C_1 \cos qy + D_1 \sin qy$ and $q^2 = \mu^2 - p^2$

$$T(x,y,t) = (A\cos px + B\sin px)(C_1 \cos qy + D_1 \sin qy)A_1 e^{-\kappa\mu^2 t}$$

$T(x,y,t) = (A\cos px + B\sin px)(C\cos qy + D\sin qy)e^{-\kappa\mu^2 t}$ Where $C = C_1 A_1,\ D = D_1 A_1$

The boundaries of the rectangle $0 \le x \le a,\ 0 \le y \le b$ are maintained at zero temperature. If at t = 0, the temperature T has prescribed value f(x,y) show that

t >0, the temperature at a point within the rectangle is given by

$T(x,y,z) = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)e^{-\kappa\mu_{mn}^2 t}$ Where

$f(m,n) = \int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)dxdy$ and $\mu_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$

Solution

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right),\quad 0 < x < a, 0 < y < b, T > 0$$

$$T(x,y,0) = f(x,y),\quad 0 < x < a,\ 0 < y < b,\ t > 0$$

$$T(0,y,t) = f(a,y,t) = 0 \qquad 0 < y < b, \quad t > 0$$

$$T(x,0,t) = f(x,b,t) = 0 \qquad 0 < x < a, \quad t > 0$$

The solution of the equation is

$T(x,y,t) = (A\cos px + B\sin px)(C\cos qy + D\sin qy)\,e^{-\kappa\mu^2 t}$ Where $\mu^2 = p^2 + q^2$

$T(x,y,t) = 0$ *and* $T(x,0,t) = 0$

$A = 0$ *and* $l = 0$ thus $T(x,y,t) = BD\sin px \sin qy e^{-\kappa\mu^2 t}$

Also the boundary condition $T(a,y,t) = 0$ and $T(x,b,t) = 0$, $\sin pa = 0$, $\sin qb = 0$, $p = \frac{m\pi}{a}$ and $q = \frac{n\pi}{a}$ respectively, where $m = 1,2,3,\cdots$ *and* $n = 1,2,3,\cdots$

Hence using the superposition principle

$$T(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{mx\pi}{a}\right)\sin\left(\frac{nx\pi}{b}\right)e^{-\kappa\mu_{mn}^2 t}\quad \text{Where } \mu_{mn}^2=p^2+q^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

Finally, the given initial condition implies

$$f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{mx\pi}{a}\right)\sin\left(\frac{nx\pi}{b}\right)$$

From this it clearly that it represents a double Fourier series particularly the sine series, to obtain we use orthogonal double Fourier series and so

$$A_{mn}=\frac{2}{a}\frac{2}{b}\int_0^a\int_0^b f(x,y)\sin\left(\frac{mx\pi}{a}\right)\sin\left(\frac{nx\pi}{b}\right)dxdy$$

Hence the require solution is

$$T(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F(m,n)\sin\left(\frac{mx\pi}{a}\right)\sin\left(\frac{nx\pi}{b}\right)e^{-\kappa\mu_{mn}^2 t}$$

$$T(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F(m,n)\sin\left(\frac{mx\pi}{a}\right)\sin\left(\frac{nx\pi}{b}\right)e^{-\kappa\mu_{mn}^2 t}\quad \text{Where}$$

$$F(m,n)=A_{mn}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{mx\pi}{a}\right)\sin\left(\frac{nx\pi}{b}\right)dxdy$$

Cauchy’s method for solving ordinary differential equations by residue theory was presented and to find the general solution for linear homogeneous differential equations with constant coefficients

Cauchy's technique for solving differential equations make use of buildup math was acquainted and with locate, the general solution for linear homogeneous differential equations with constant coefficients

$$a_0y^n+a_1y^{n-1}+\cdots+a_{n-1}y'+a_ny=0 \quad \textbf{(1)}$$

Where $a_0=1$ *and* $a_j$'s , $j=1,2,3,\cdots$ are given constant s.

**Theorem 1**

Consider the differential equations with constant coefficients.

$$y^n+a_1y^{n-1}+\cdots+a_{n-1}y'+a_ny=0 \quad \textbf{(2)}$$

Let f be an arbitrary function of the complex variable z, whose zeros do not coincide with the zeros of the polynomial

$$g(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_{n-1}z+a_n \quad \textbf{(3)}$$

Then the general solution of (4.2.2) is given by

$$y(x) = \sum res\left[\frac{f(x)e^{zx}}{g(z)}\right]$$

**(4)**

We now show that (4) is a solution of the homogeneous differential equation (2). We assume that

$$y = \sum res\left[\frac{f(x)e^{zx}}{g(z)}\right]$$

Then

$$y' = \sum res\left[\frac{f(x)e^{zx}}{g(z)}.z\right]$$

$$y'' = \sum res\left[\frac{f(x)e^{zx}}{g(z)}.z^2\right]$$

$$\vdots$$

$$y^k = \sum res\left[\frac{f(x)e^{zx}}{g(z)}.z^k\right] \quad \text{(k =1, 2,3...n)}$$

Hence

$$y^n + a_1 y^{n-1} + \cdots + a_{n-1}y' + a_n y \qquad \textbf{(5)}$$

$$= \sum \operatorname{Res}\left[\frac{f(z)e^{zx}}{g(z)}z^n\right] + a_1 \sum \operatorname{Res}\left[\frac{f(z)e^{zx}}{g(z)}z^{n-1}\right] + \cdots + a_{n-1}\sum \operatorname{Res}\left[\frac{f(z)e^{zx}}{g(z)}z\right] + a_n \sum \operatorname{Res}\left[\frac{f(z)e^{zx}}{g(z)}\right]$$

$$= \sum \operatorname{Res}\left[\frac{f(z)}{g(z)}e^{zx}\left(z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n\right)\right] = \sum \operatorname{Res}\left[\frac{f(z)}{g(z)}e^{zx}g(z)\right]$$

$$= \sum \operatorname{Res}\left[f(z)e^{zx}\right] = 0$$

Since f(z) is analytic. Thus, (4) is indeed a solution of (2), i.e. (4) is a general solution

**3.**

**4. RESULT**

**Problem 1** : solve the equation $u_t = c^2\left(u_{xx} + u_{yy}\right) 0 < x < \pi, \quad 0 < y < \pi \ t > 0$

$u(x, y, 0) = x(\pi - x)y(\pi - y) \quad 0 < x < \pi, \ 0 < y < \pi$

$u_x(0,y,t)=u_x(\pi,y,t)=0$, $u_y(x,0,t)=u_y(x,\pi,t)=0$

$\frac{T'}{c^2T}=\frac{X''}{X}+\frac{Y''}{Y}$ and $\frac{X''}{X}=R$ *and* $\frac{Y''}{Y}=P \Rightarrow X''-XR=0$ *and* $Y''-PY=0$

CASE 1: $R=\lambda^2>0$

$$X''-\lambda^2X=0 \quad \Rightarrow X(x)=c_1e^{\lambda x}+c_2e^{-\lambda x}$$

$X'(x)=\lambda c_1e^{\lambda x}-\lambda c_2e^{-\lambda x}$ $X(0)=0$ *and* $X(\pi)=0$ the solution is trivial.

CASE 2: R = 0

$X''=0 \quad \Rightarrow c_1x+c_2$ and $X(0)=0$, $X(\pi)=0$ the solution is trivial

CASE 3: $R=-\lambda^2<0$

$X(x)=c_1\cos\lambda x+c_2\sin\lambda x$

$X'(x)=-\lambda c_1\sin\lambda x+\lambda c_2\cos\lambda x$ $X'(0)=0 \Rightarrow c_2=0$ *and* $X'(\pi)=0 \Rightarrow \lambda=n$

$X_n(x)=c_n\cos nx$

$Y''-PY=0$

CASE 1: $p=\lambda^2>0$

$Y''-\lambda^2Y=0$ $Y(y)=c_1e^{\lambda y}+c_2e^{-\lambda y}$ $\Rightarrow Y(y)=\lambda c_1e^{\lambda y}-\lambda c_2e^{-\lambda y}$

$Y'(y)=0$ *and* $Y(\pi)=0$ the solution is trivial

CASE 2: $p=0$

$Y''=0 \quad \Rightarrow Y(y)=c_1y+c_2$ and $Y'(y)=0$ *and* $Y(\pi)=0$ the solution is **trivial**

CASE 3: $R=-\lambda^2<0$

$Y(y)=c_1\cos\lambda y+c_2\sin\lambda y$

$Y'(y)=-\lambda c_1\sin\lambda y+\lambda c_2\cos\lambda y$ $Y'(0)=0 \Rightarrow c_2=0,$ $Y'(\pi)=0 \Rightarrow \lambda=m$

$Y_m(y)=c_m\cos my$ it follows that

$$\frac{T'}{c^2T}=n^2+m^2 \qquad \Rightarrow T(t)=e^{-c^2t(n^2+m^2)}$$

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{nm}\cos nx\cos my e^{-c^2t(n^2+m^2)}$$

$$u(x,y,0)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{nm}\cos nx\cos my$$

$c_{nm}=\dfrac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi}x(\pi-x)y(\pi-y)\cos nx\cos my\,dxdy=\dfrac{4(-1)^{n+m+2}}{n^2m^2}$ using double Fourier series

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{4(-1)^{n+m+2}}{n^2m^2}\cos nx\cos my e^{-c^2t(n^2+m^2)}$$

**Problem 2** : solve the equation $u_t=(u_{xx}+u_{yy})\,0<x<\pi,\quad 0<y<\pi\ t>0$

$u(x,y,0)=x\quad 0<x<\pi,\ 0<y<\pi$

$u_x(0,y,t)=u_x(\pi,y,t)=0,\quad u_y(x,0,t)=u_y(x,\pi,t)=0$

$\dfrac{T'}{T}=\dfrac{X''}{X}+\dfrac{Y''}{Y}$ and $\dfrac{X''}{X}=R\ and\quad \dfrac{Y''}{Y}=P\Rightarrow X''-XR=0\ and\ \ Y''-PY=0$

CASE 1: $R=\lambda^2>0$

$$X''-\lambda^2X=0\quad\Rightarrow X(x)=c_1e^{\lambda x}+c_2e^{-\lambda x}$$

$X'(x)=\lambda c_1e^{\lambda x}-\lambda c_2e^{-\lambda x}\quad X(0)=0\ \ and\quad X(\pi)=0$ the solution is trivial.

CASE 2: R = 0

$X''=0\quad\Rightarrow c_1x+c_2$ and $X(0)=0,\quad X(\pi)=0$ the solution is trivial

CASE 3: $R=-\lambda^2<0$

$X(x)=c_1\cos\lambda x+c_2\sin\lambda x$

$X'(x)=-\lambda c_1\sin\lambda x+\lambda c_2\cos\lambda x\quad X'(0)=0\ \ \Rightarrow c_2=0\ \ and\ \ X'(\pi)=0\ \Rightarrow\lambda=n$

$X_n(x)=c_n\cos nx$

$Y''-PY=0$

CASE 1: $p=\lambda^2>0$

$Y''-\lambda^2Y=0\quad Y(y)=c_1e^{\lambda y}+c_2e^{-\lambda y}\qquad\qquad\Rightarrow Y(y)=\lambda c_1e^{\lambda y}-\lambda c_2e^{-\lambda y}$

$Y'(y)=0\ \ and\quad Y(\pi)=0$ the solution is trivial

CASE 2: $p = 0$

$Y'' = 0 \quad \Rightarrow Y(y) = c_1 y + c_2$ and $Y'(y) = 0$ *and* $Y(\pi) = 0$ the solution is **trivial**

CASE 3: $R = -\lambda^2 < 0$

$$Y(y) = c_1 \cos \lambda y + c_2 \sin \lambda y$$

$$Y'(y) = -\lambda c_1 \sin \lambda y + \lambda c_2 \cos \lambda y \quad Y'(0) = 0 \quad \Rightarrow c_2 = 0, \quad Y'(\pi) = 0 \Rightarrow \lambda = m$$

$Y_m(y) = c_m \cos my$ it follows that

$$\frac{T'}{T} = n^2 + m^2 \qquad \Rightarrow T(t) = e^{-t(n^2+m^2)}$$

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \cos nx \cos mye^{-t(n^2+m^2)}$$

$$u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \cos nx \cos my$$

$c_{nm} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} y \cos mydxdy = \frac{4}{m^2} \left[(-1)^m - 1\right]$ using double Fourier series

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4\left((-1)^m - 1\right)}{m^2} \cos nx \cos mye^{-t(n^2+m^2)}$$

## 5. Conclusion

This paper has aimed to construct an analytic solution to the two-dimensional heat equation problems with Neumann boundary condition using separation of variable of partial differential equations and double Fourier series, *then residue theory was employed to solve the homogenous differential equations*. On Solving an analytic solution to the two-dimensional heat equation problems with Neumann boundary condition using separation of variable of partial differential equations and double Fourier series gave the exact analytical solution of the cosine function. However, *In case one (1) , two (2) and three (3) the result obtained using residue theory is the same as the other method of solving ordinary differential equation, case one (1) and two (2) the analytic solutions are trivial, while case three (3) the solutions is not* .It is worth mentioning that the technique and ideas presented in this paper can be extended to finding the contact problems encountered in sciences, engineering, and Technology.

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