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Application of Residue Calculus on Second Order Linear Homogeneous Differential Equation

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Abstract: *The focus of this study is to show the application of residue calculus of second order linear Homogeneous differential equations. Cauchy method was introduced to solve second order linear homogeneous differential equation. In comparing the method of finding the Cauchy's method for solving ordinary second order homogeneous differential and the order method both result are the same. The Cauchy's residue method is more direct, precisely, efficient, and time-saving. The result obtained shows that it has no complex solution. However both result are real and have their applications in electrical and mechanical engineering systems. Electrical application can be used in an electrical circuit such as Resistor, inductor and capacitor and the Mechanical can be used to describe acceleration, velocity and displacement. In short, the result has application in the field of engineering particularly Electrical and mechanical engineering.*

Keywords: *Second order differential. Residue calculus, Homogeneous Differential Equation*

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1. INTRODUCTION

In complex analysis, a field in mathematics, the residue theorem, sometimes called Cauchy's residue theorem(one of many things named after Augustin-Louis Cauchy), is a powerful tool to evaluate line integral of analytic function over closed curves; it can be used to compute real integral as well. It generalizes the Cauchy integral theorem and Cauchy's integral formula. From a geometrical perspective, it is a special case of the generalised stokes theorem. The analytical evaluation of a general contour integral with integral f(z) depends for its success on what are called the residue at the poles of f(z).The residue of a function f(z) at a pole of defined in terms of a special series expansion of f(z) about the pole called a Laurent series. The Laurent series represents an extension of the conventional Taylor series that is no longer applicable expansion of f (z) is required about a singular point. Various ways of obtaining Laurent series are described, and it is shown how a contour integral is related to the residue of the integrant f (z) that lie either inside or on the contour of integration different types of contour integral are evaluated and integration around a branch point of f(z) is considered. The mathematical field of complex analysis, contour integration is a method of evaluating certain integrals along path in the complex plane. Contour integration is closely related to calculus of residues a methodology of complex analysis. One use for contour integrals is the evaluation of integral along the real line that is not readily found by the using only real variable methods.

This work shows the application of residue calculus on second order homogeneous differential equation. Residue calculus is one of the most important notions of mathematics which find it application in all fields, especially in science and technology. The objective of this paper is to introduce Cauchy method for solving linear second order homogeneous differential equations with constant coefficients and its applications.

2. LITERATURE

An equation of the form $x''+Px'+Qx=0$ where P and Q are function of x, is said to be homogeneous differential equation, if it is not equal to zero is said to be non-homogeneous differential equation. Second order differential equations are classical methods which have wide area of applications in the field of science and technology. Especially in Engineering discipline. Generally, the differential equation may be ordinary or partial differential equation. [5]

Proposed was used On-chip Tunable second order differential equation solver Base on a silicon Photonic Mode-split micro-resonator with Tunable coefficients and system demonstration using the fabricated device is carried out for 10-Gb/s Gaussian and supper Gaussian in put pulse. The experimental results are in good agreement with theoretical prediction of the solution. [9]

Physics laws are generally written in the form of differential equations, science and technology are use differential equation the main concerned of differential equations are one the most important part of language of science and technology. [6]

Application of differential transform method for solving differential and integral equations was used to apply in area of Engineering and science using differential transform method was employed to solve Volterra integral equation of second kind. Taylor series polynomial or expansion was also used to construct analytical approximate solution of initial value problem. Differential transform method has been successfully use for finding solution of non-linear system of volterra integral equation and its powerful tools technique for obtained exact solution.[10]

Second order differential equation arises for the charges on a capacitor in unpowered RLC electrical circuit or freely oscillating frictional mass on a spring or for a damping pendulum.

Application of second differential equation was recently discuss [7], in the context, Schrödinger equation were discuss and condition under which the confluent, biconfluent and the generally, exact solution are transcendental functions and was a recursive ring and the Schrödinger equation where is extremely use full to have at one’s disposal some. Application of the results to Schrödinger’s equation was discussed and conditions under which the confluent, biconfluent, and the general Heun equation yield polynomial solution are explicitly given. [1]

Determining dynamic market Equilibrium price function was used to obtained the equilibrium price function over in dynamic market, by observing the price changes and changes in the level of rising prices. Second order non-homogeneous differential equation was used to determining Dynamic market Equilibrium price functions over a time by considering the price change and change in the level of rising price. [8]

Most of the cited literature they did not apply the method of Cauchy to solve second order ordinary differential equation, in other to solve the second order homogeneous differential equation. But they applied various methods to solve it.

3. PROCEDURE

GENERAL SOLUTION OF ORDINARY DIFFERENTIONAL EQAUTIONS

Here we introduce Cauchy's method for solving ordinary differential equations using residue. Specially, we will find the general solution for linear homogeneous differential equations with constant coefficients

$$a_0 y^n + a_1 y^{n-1} + \cdots + a_{n-1} y' + a_n y = 0 \qquad \textbf{(1)}$$

Where $a_0 = 1 \ and \ a_j{}^{'s}, \ j = 1,2,3,\cdots$ are given constant s.

Theorem 1

Consider the differential equations with constant coefficients.

$$y^n + a_1 y^{n-1} + \cdots + a_{n-1} y' + a_n y = 0 \qquad \textbf{(2)}$$

Let *f* be an arbitrary function of the complex variable *z*, whose zeros do not coincide with the zeros of the polynomial

$$g(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n \qquad \textbf{(3)}$$

Then the general solution of (4.2.2) is given by

$$y(x) = \sum res\left[\frac{f(x)e^{zx}}{g(z)}\right] \qquad \textbf{(4)}$$

We now show that (4) is a solution of the homogeneous differential equation (2). We assume that

$$y = \sum res\left[\frac{f(x)e^{zx}}{g(z)}\right]$$

Then

$$y' = \sum res\left[\frac{f(x)e^{zx}}{g(z)}.z\right]$$

$$y'' = \sum res\left[\frac{f(x)e^{zx}}{g(z)}.z^2\right]$$

$$\vdots$$

$$y^k = \sum res\left[\frac{f(x)e^{zx}}{g(z)}.z^k\right] \quad (k \ =1, 2,3\ldots n)$$

Hence

$$y^n + a_1 y^{n-1} + \cdots + a_{n-1}y' + a_n y \tag{5}$$

$$= \sum \mathrm{Re}\, s\left[\frac{f(z)e^{zx}}{g(z)} z^n\right] + a_1 \sum \mathrm{Re}\, s\left[\frac{f(z)e^{zx}}{g(z)} z^{n-1}\right] + \cdots + a_{n-1}\sum \mathrm{Re}\, s\left[\frac{f(z)e^{zx}}{g(z)} z\right] + a_n \sum \mathrm{Re}\, s\left[\frac{f(z)e^{zx}}{g(z)}\right]$$

$$= \sum \mathrm{Re}\, s\left[\frac{f(z)}{g(z)} e^{zx}\left(z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n\right)\right] = \sum \mathrm{Re}\, s\left[\frac{f(z)}{g(z)} e^{zx} g(z)\right]$$

$$= \sum \mathrm{Re}\, s\left[f(z)e^{zx}\right] = 0$$

Since f(z) is analytic. Thus, (4) is indeed a solution of (2), i.e. (4) is a general solution

4. RESULT

Problem 3.1: - The case of distinct real roots. We want to find the general solution of the differential equations

$$y'' - 7y' + 12 = 0$$

Let f(z) be any arbitrary entire function whose zero are 3 and 4

Let $g(z) = z^2 - 7z + 12$. Then $g(z) = (z-3)(z-4)$

Clearly the zeros of f (z) do not coincide with the zero of g(z). We know the general solution is given by

$$y = \sum \operatorname{Re} s\left[\frac{f(z)}{g(z)}\right]$$ By **(4)**

Where the summation is take over z = 3 and z = 4. So we have

$$y = \operatorname{Re} s\left(\frac{f(z)}{g(z)}e^{zx};3\right) + \operatorname{Re} s\left(\frac{f(z)}{g(z)}e^{zx};4\right)$$

$$= \lim_{z\to 3}(z-3)\frac{f(z)e^{zx}}{(z-3)(z-4)} + \lim_{z\to 4}(z-4)\frac{f(z)e^{zx}}{(z-3)(z-4)}$$

$$= \lim_{z\to 3}\frac{f(z)e^{zx}}{(z-4)} + \lim_{z\to 4}\frac{f(z)e^{zx}}{(z-3)} = -f(3)e^{3x} + f(4)e^{4x}$$

Let $let\ \ c_1 = -f(3)\ \ and\ \ c_2 = f(4)$

Hence the general solution of differential equation is

$$y = c_1 e^{3x} + c_2 e^{4x}$$

Problem 3.2: - the case of repeated real roots. It is our objective to find the general solution of the differential equations.

$$y'' - 6y' + 9y = 0$$

Let f(z) be an arbitrary entire function whose zero's do not include 3, and let

$g(z) = z^2 - 6z + 9$ Then $g(z) = (z-3)^2$

The general solution is given by

$$y = \sum Res\left[\frac{f(z)}{g(z)}e^{zx}\right]$$

Then we have

$$y = \sum Res\left[\frac{f(z)}{(z-3)^2}e^{zx},3\right]$$

$$= \lim_{z\to 3}\frac{1}{(2-1)!}\frac{d}{dz}\left[(z-3)^2\frac{f(z)}{(z-3)^2}e^{zx}\right] = \lim_{z\to 3}\frac{d}{dz}\left[f(z)e^{zx}\right]$$

$$= \lim_{z\to 3}\left[f(z)xe^{zx} + f'(z)e^{zx}\right] = \left[f(3)xe^{3x} + f'(3)e^{3x}\right]$$

$let \quad f'(3) = c_1 \quad and \quad f(3) = c_2$

Thus, we have obtain the general solution,

$$y = c_1e^{3x} + c_2xe^{3x}$$

Problem 3.3

The equation $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{Lc}i = 0$

Represent a current i flowing in an electrical circuit containing resistance R and inductance L and capacitance C connected in series. If R = 200 ohms, L = 0.20 henry and $C = 20\times 10^{-6}$ *farads* , solve the equation for i given the boundary conditions that when t = 0, i = 0 and $\frac{di}{dt} = 100$

Solution

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{Lc}i = 0$$

$$\frac{d^2i}{dt^2} + \frac{200}{0.20}\frac{di}{dt} + \frac{1}{0.20\times 20\times 10^{-6}}i = 0$$

$$\frac{d^2i}{dt^2} + 1000\frac{di}{dt} + 250000i = 0$$

$$g(z) = z^2 + 1000z + 250000 = 0$$

$$z = \frac{-1000 \pm \sqrt{1000000 - 1000000}}{2} = \frac{-100}{2} = -500$$

$z = -500$ Of order 2

Res (-500)

$$= \lim_{z \to -500} \frac{1}{(2-1)!} \frac{d}{dz} \left[\frac{(z+500)^2 f(z) e^{zt}}{(z+500)^2} \right] = \lim_{z \to -500} \frac{1}{(2-1)!} \frac{d}{dz} \left[f(z) e^{zt} \right]$$

$$= \lim_{z \to -500} \left[f(z) te^{zt} + f'(z) e^{zt} \right]$$

$$i = f(-500) te^{-500t} + f'(-500) e^{-500t}$$

$$i = c_1 te^{-500t} + c_2 e^{-500t}$$

When t =0 , i = 0 and c = 0

$$\frac{di}{dt} = -500 c_1 te^{-500t} + c_1 e^{-500t} - 500 c_2 e^{-500t}$$

$$100 = c_1 - 500 c_2 \ \Rightarrow c_1 = 100$$

$$i = 100 te^{-500t}$$

Problem 3.4

$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} \frac{1}{C} i = 0$ is an equation representing current i in an electrical circuit. If inductance L is 0.25 henry, capacitances C is 29.76×10^{-6} farads and R is 250 ohms, solve the equation for i given the boundary condition that when t = 0, i = 0 and $\frac{di}{dt} = 34$

Solution

$$0.25 \frac{d^2 i}{dt^2} + 250 \frac{di}{dt} + \frac{1}{29.76 \times 10^{-6}} = 0$$

$$0.25\frac{d^2 i}{dt^2}+250\frac{di}{dt}+3360.215054=0$$

$$f(z)=Lz^2+Rz+\frac{1}{C}$$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-R\pm\sqrt{R^2-\frac{4L}{C}}}{2a}=\frac{-250\pm\sqrt{250^2-\frac{4(0.25)}{29.76\times10^{-6}}}}{2(.25)}$$

$$z=\frac{-250\pm\sqrt{28897.84946}}{0.5}=\frac{-250\pm169.99}{0.5}=\frac{-250\pm170}{0.5}$$

$$z=\frac{-250+170}{0.5}\ or\ z=\frac{-250-170}{0.5}$$

$$z=\frac{-80}{0.5}\ or\ z=\frac{-420}{0.5}$$

$$z=-160 or\ z=-840$$

Res (-160) + Res (-840)

$$i=\operatorname{Re}s\left(\frac{f(z)}{g(z)}e^{zt},-160\right)+\operatorname{Re}s\left(\frac{f(z)}{g(z)}e^{zt},-840\right)$$

$$i=\lim_{z\to-160}\frac{(z+160)f(z)e^{zt}}{(z+160)(z+840)}+\lim_{z\to-160}\frac{(z+840)f(z)e^{zt}}{(z+840)(z+160)}$$

$$i = \frac{f(-160)e^{-160t}}{680} + \frac{f(z)e^{-840t}}{-680}$$

$$i = c_1 e^{-160t} + c_2 e^{-840t}$$

$$0 = c_1 + c_2 \Rightarrow c_2 = -c_1$$

$$\frac{di}{dt} = -160c_1 e^{-160t} - 840c_2 e^{-840t}$$

$$34 = -160c_1 - 840c_2$$

$$34 = 680c_1$$

$$c_1 = \frac{1}{20}$$

$$but \quad c_2 = -c_1$$

$$c_2 = \frac{-1}{20}$$

$$i = \frac{1}{20}\left(e^{-160t} - e^{-840t}\right)$$

Problem 3.5

The displacement s of a body in a damped mechanical system, with no external forces satisfies the following differential equation

$$2\frac{d^2 s}{dt^2} + 6\frac{ds}{dt} + 4.5s = 0$$

Where t represent time, it initially condition, when t = 0 , s = 0 and $\frac{ds}{dt} = 4$ solve differential equation of S in terms of t.

Solution

$$2\frac{d^2 s}{dt^2} + 6\frac{ds}{dt} + 4.5s = 0$$

$$g(z) = 2z^2 + 6z + 4.5$$

$$z = \frac{-6 \pm \sqrt{36-36}}{4} = \frac{-3}{4}$$

Res (-3) of order 2

$$s = \lim_{z \to -\frac{3}{2}} \frac{d}{dz}\left[\frac{(z+3)^2 f(z) e^{zt}}{(z+3)}\right] = \lim_{z \to -\frac{3}{2}} \frac{d}{dz}\left[f(z) e^{zt}\right]$$

$$s = \lim_{z \to \frac{-3}{2}} \left[f(z) te^{zt} + f'(z) e^{zt}\right]$$

$$s = f\left(\frac{-3}{2}\right) te^{\frac{-3}{2}t} + f'\left(-\frac{3}{2}\right) e^{\frac{-3}{2}t}$$

$$s = c_1 te^{-\frac{3}{2}t} + c_2 e^{-\frac{3}{2}t}$$

When t = 0 and s =0 then $c_2 = 0$

$$\frac{ds}{dt} = -\frac{3}{2} c_1 te^{-3t} + c_1 e^{-3t} - \frac{3}{2} c_2 e^{-3t}$$

$$4 = c_1 - \frac{3}{2} c_2$$

$$c_1 = 4$$

$$s = 4te^{-3t}$$

5. CONCLUSION AND RECOMMENDATION

In comparing the method of finding the Cauchy's method for solving ordinary second order homogeneous differential and the order method both result are the same. The Cauchy's residue method is more direct, precisely, efficient, and time-saving. The result obtained shows that it has no complex solution. However both result are real and have their applications in electrical and mechanical engineering systems. Electrical application can be used in an electrical circuit such as Resistor, inductor and capacitor and the Mechanical can be used to describe acceleration, velocity and displacement. In short, the result have application in the field of engineering particularly Electrical and mechanical engineering.

It has been recommended that the method should be applied in solving problems in engineering and other related fields.

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